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Singularity considerations in membrane, plate and shell behaviors

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Abstract

This paper describes three types of situations in structural analysis where singularities at points can greatly influence the global behavior of the configuration: (1) concentrated forces acting upon flat or curved membranes, (2) concentrated moments acting upon plates or shells, and (3) sharp corner sigularities in plates and shells. These singularities may have strong effects upon static or dynamic deflections, free vibration frequencies and buckling loads. It is shown that the concentrated forces acting upon flat or curved membranes, or concentrated moments acting upon plates and shells, are improper models, and that correct theoretical analysis indicates that they are meaningless. Examples of sharp corners discussed are (1) the re-entrant corner of a cantilever skew plate, (2) a free circular plate with a V-notch, and (3) the obtuse corners of a simply supported parallelogram plate.

Keywords: Membranes; Plates; Shells; Vibration

1. Introduction

In four decades of research on membranes, plates and shells, the writer has encountered numerous situations where infinite displacements, slopes, forces and/or bending and twisting moments arise. Some of the published papers were studied in connection with the plate vibration monograph (Leissa, 1969), which contains 500 references, and with the shell vibration monograph (Leissa, 1973), including 1000 references. An additional 1000 plate vibration references were included in a series of subsequent review articles. A summary of the laminated composite plate and theory literature on buckling and post-buckling examined 400 references (Leissa, 1985, 1987). And at least another 1000 papers dealing with membranes, plates and shells have been looked through for other circumstances.

The infinite quantities arise in the mathematical solutions of membrane, plate and shell problems, and these are termed out "singularities". They are caused by various idealizations used to simplify representation of physical problems, such as concentrated forces and moments, discontinuities in edge conditions, or sharp corners. Typically, they are the consequences of theories and idealizations which have existed for a century

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or more, in order to simplify the mathematical solutions. Although current large computational capabilities permits one to eliminate some of these singularities, doing so is typically not easy. That is, the more correct idealizations may require enormous computational efforts.

Using classical theories for membrane, plate and shell displacements, one should be aware of their shortcomings and limitations. This is especially true for the singularities which arise. If they are not considered properly, then highly inaccurate, or even meaningless, results can arise.

In this paper, the writer shares some of his thoughts on three types of singularities which may arise, and are sometimes not considered properly by researchers and users of structural mechanics. These include (1) concentrated forces acting upon membranes, (2) concentrated moments acting upon plates and shells, and (3) sharp corners, both interior and exterior, used in models of plates and shells. Emphasis is given to the pitfalls which should be avoided.

2. Concentrated forces and reactions on membranes

The classical equation of transverse motion for a perfectly flexible, planar membrane stretched uniformly in all directions is

$$T\nabla^2 w + p = \rho h \frac{\partial^2 w}{\partial t^2}, \tag{1}$$

where T is the uniform inplane tension, ∇^2, the Laplacian operator, p, a distributed transverse pressure, t, time, and ρ and h are the mass density and thickness, respectively, of the membrane. If the right-hand side of Eq. (1) is zero, one has a statically loaded membrane, if $p = 0$, the free vibration problem is described, and if neither is zero, one has the dynamic response situation, including forced vibration.

All is well for these problems if T is sufficiently large, the transverse displacement, w, is sufficiently small (so that T does not change significantly), and the membrane slope is small. These assumptions are required in order to arrive at the linear form of Eq. (1). But a serious flaw arises if p is taken to be a concentrated (i.e., point) force, instead of a distributed one. This shortcoming seems to be widely recognized among analysts considering statically applied forces, but often ignored by those solving dynamic problems, especially when the concentrated force is the result of a point constraint.

Let us first review the well-known result for the static problem. Fig. 1(a) is a three-dimensional view of a circular membrane of outer radius a, supposedly subjected to a point force, P, at its center. A cross-section of the membrane taken at an arbitrary radius r is depicted in Fig. 1(b).

Because T is force/length, summing forces in the transverse direction yields

$$P - 2\pi r T \sin\phi = 0. \tag{2}$$

Assuming small slopes, $\sin\phi \approx \tan\phi = -\partial w/\partial r$. Substituting this into Eq. (2), and solving for the slope,

$$\frac{\mathrm{d}w}{\mathrm{d}r} = -\left(\frac{P}{2\pi T}\right)\frac{1}{r}. \tag{3}$$

Integrating and applying the boundary condition $w(a) = 0$ results in

$$w = -\frac{P}{2\pi T}\ln\left(\frac{r}{a}\right). \tag{4}$$

Evaluating this at $r = 0$, one find that

$$w(0) = \infty, \qquad \frac{\mathrm{d}w}{\mathrm{d}r}(0) = -\infty \tag{5}$$

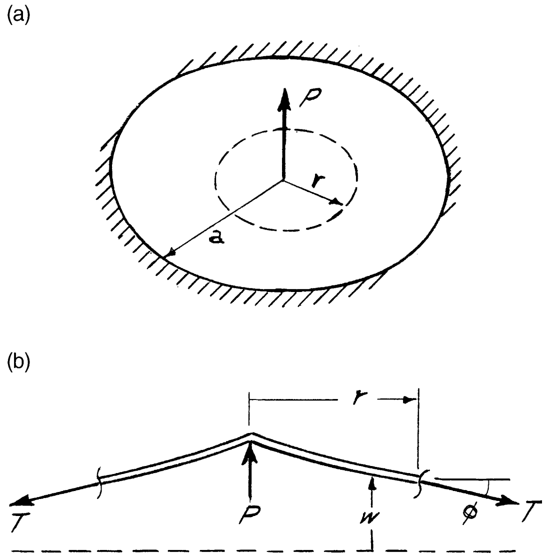

Fig. 1. Circular membrane subjected to a concentrated force at its center: (a) three-dimensional overall view and (b) cross-section in the vicinity of the force.

no matter how small is P. Thus, according to the linear theory, the membrane has *no resistance to the concentrated force*. The smallest force results in an infinite deflection.

One can approach this situation less directly by first applying P to a small, rigid disk at the membrane center (Fig. 2), and then letting the disk radius, b, approach zero. The force P enters through a boundary condition at $r = b$ (the disk radius), and the membrane deflection is determined by the special case of Eq. (1) with $p = 0$; i.e., $\nabla^2 w = 0$. Its axisymmetric solution in polar coordinates is

$$w = C_1 \ln r + C_2, \tag{6}$$

where C_1 and C_2 are constants of integration. The boundary conditions are $w(a) = 0$, and for small slopes, at $r = b$, summing forces on the rigid disk,

$$2\pi b T \frac{\mathrm{d}w}{\mathrm{d}r}(b) + P = 0. \tag{7}$$

Applying the two boundary conditions, and solving for C_1 and C_2, one obtains again Eq. (4) which, of course, now applies only to $r \geqslant b$. As b becomes small, the same undesirable infinite values of deflection and slope arise for any finite P.

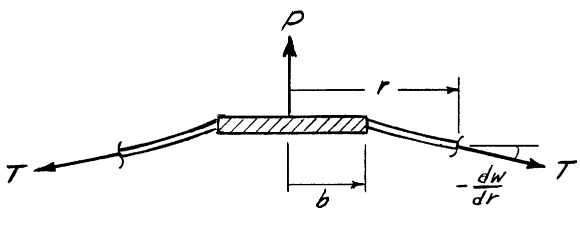

Fig. 2. Cross-section of a circular membrane, with a transverse force (P) applied to a rigid central disk of radius b.

The membranes described above had circular boundaries, and the point force was applied at their centers. However, the force could have acted at any point with the same fundamental result – finite forces cause infinite displacements where they are applied, according to the classical, linear theory. Nor does this conclusion depend upon the shape of the boundary. It can be elliptical, rectangular, or arbitrary. In all cases, the membranes supply no resistance to the forces at their points of application.

The concentrated force could also be, supposedly, a reactive one. To approach such a situation clearly, consider first an *annular* membrane fixed at its outer ($r = a$) and inner ($r = b$) circular boundaries, subjected to uniform pressure (p_0), as described by Fig. 3. Eq. (1) then reduces to its equilibrium form,

$$\nabla^2 w = -\frac{p_0}{T}. \tag{8}$$

For uniform pressure, the static deflection is axisymmetric, so that

$$\nabla^2 = \frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}. \tag{9}$$

The solution to Eq. (8) is

$$w = C_1 \ln r + C_2 - \frac{p_0 r^2}{4T}. \tag{10}$$

Applying the boundary conditions $w(a) = w(b) = 0$ yields

$$C_1 = \frac{p_0 a^2}{4T}\frac{(b^2/a^2 - 1)}{\ln(b/a)}, \quad C_2 = \frac{p_0 a^2}{4T}\left[1 - \frac{(b^2/a^2 - 1)\ln a}{\ln(b/a)}\right]. \tag{11}$$

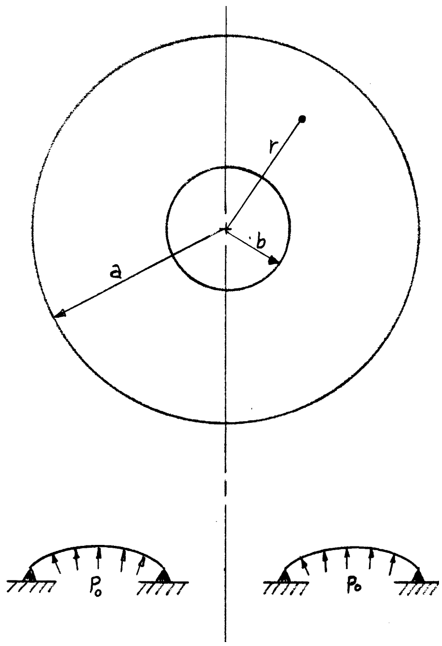

Fig. 3. Annular membrane subjected to uniform pressure.

The reactive force at the inner boundary ($r = b$) is

$$F = \left[2\pi r T \frac{\mathrm{d}w}{\mathrm{d}r}\right]_{r=b} \tag{12a}$$

$$= 2\pi T\left(\frac{p_0 a^2}{2T}\right)\left(\frac{b^2/a^2 - 1}{\ln(b/a)} - \frac{b^2}{a^2}\right). \tag{12b}$$

Taking the limit of the right-hand side as $b/a \to 0$, shows that $F \to 0$ as the inner circular boundary reduces to a point. Thus, adding a point constraint to a membrane already supported on a circular boundary does not help support the pressure loading. It is all supported by the outer boundary. This is verified by evaluating Eq. (12a) at $r = a$, and taking its limit as $b \to 0$, which yields $F = \pi a^2 p_0$ at the outer boundary.

The example immediately preceding should hardly be necessary. If a membrane can render no resistance to an applied point force, then a "support point" can add no stiffness to the system. Interestingly, the solution to the linear problem does continue to enforce zero displacement on the inner circular boundary as b approaches zero, but the reactive force disappears.

In the published literature, there are a few research papers which purport to solve free vibration problems for flat membranes which are constrained, in part, at points. The solution procedures used are not exact, and therefore, the problems being solved are only approximations to the claimed physical situations. That is, the resulting constraint forces are distributed over small, but finite, areas, or the differential equation (1) is only approximately satisfied – a residual pressure results which should be zero, but is not, and may have singularities.

To demonstrate the aspect described above, consider the free vibrations of an annular membrane fixed at its outer and inner boundaries ($r = a$ and b). For free vibrations $p = 0$ in Eq. (1). The well-known solution to Eq. (1) is

$$w = [A_n J_n(kr) + B_n Y_n(kr)]\cos n\theta \sin \omega t, \tag{13}$$

where A_n and B_n are undetermined constant coefficients, J_n and Y_n, Bessel functions of the first and second kinds, respectively, of order n; $k = \rho h\omega^2/T$, $n = 0, 1, \ldots, \infty$, θ, the circumferential angle, and ω, a free vibration frequency. The displacement function $w(r, \theta, t)$ given by Eq. (13) represents all possible free vibrational modes for membranes fixed entirely around their peripheries (i.e., $0 \leqslant \theta \leqslant 2\pi$).

Substituting Eq. (13) into the boundary conditions $w(a, \theta, t) = w(b, \theta, t) = 0$ yields two homogeneous algebraic equations in the coefficients A_n and B_n. For a nontrivial solution, one sets the determinant of the coefficient matrix equal to zero, resulting in the frequency equation

$$J_n(\lambda)Y_n\left(\frac{b}{a}\lambda\right) - J_n\left(\frac{b}{a}\lambda\right)Y_n(\lambda) = 0, \tag{14}$$

where $\lambda = ka$ are the desired eigenvalues, which are the nondimensional frequency parameters $\omega a(\rho h/T)^{1/2}$.

Table 1 lists the first three frequencies (corresponding to 0, 1 and 2 interior nodal circles in the mode shapes) for the *axisymmetric* ($n = 0$) modes of annular membranes having various *b*/*a* ratios, especially for small *b*/*a*. One sees that, as *b*/*a* approaches zero, the lowest λ approaches 2.405, which is the fundamental frequency for the well-known case of a circular membrane *not* having an interior support. And the next two frequencies approach those of the next two axisymmetric mode frequencies (5.520 and 8.654) of the classical circular membrane. Thus, again, the point support adds no stiffness to the system. One would not expect the frequencies for the *nonaxisymmetric* modes ($n = 1, 2, \ldots,$) to be affected by a central point support in any case, for the diametral node lines fall upon the support. But a careful (i.e. accurate) solution of the problem of the membrane supported at a *noncentral* point would also show all frequencies to be unaffected by the support point. To generalize further, as was done previously for the static problem, adding point supports to a membrane of *any* shape will not change its frequencies (again, according to the classical theory).

Table 1
Nondimensional frequencies $\lambda = \omega a(\rho h/T)^{1/2}$ for the axisymmetric ($n = 0$) modes of annular membranes

b/a	Mode number		
	1	2	3
0.40	5.183	10.443	15.688
0.20	3.816	7.786	11.732
0.10	3.314	6.858	10.377
0.02	2.884	6.136	9.376
0	2.405	5.520	8.654

By similar arguments, one sees that a transverse spring attached to a membrane does nothing to stiffen it, and that a particle of mass attached does nothing to change the inertia of the system. Both must be distributed over finite lengths, or areas, to influence the membrane. And a concentrated exciting force, varying sinusoidally in time, will elicit no response at all from the membrane, according to classical theory.

The examples above show that the classical theory for planar membranes clearly is inappropriate when applied or reactive transverse forces are supposedly applied at points or, as may be inferred, when their areas or circumferences of application become very small. Improved, nonlinear theories may be developed which account for large slopes and material stretching (causing T which depend upon w). However, they also could not deal with a point force because, as the zone of application diminishes in size, T would have to become infinite to generate a finite transverse force. Such theories, although difficult to solve, would be much better than the classical, linear theory for small (but finite) zones of application.

Finally, consider a nonplanar membrane as, for example, a balloon. Such configurations may be termed *shells* without bending stiffness. Indeed, much has been written about membrane theories of shells, and solutions of the resulting fourth-order system of differential equations which apply when the bending and twisting moments are negligible. It is also known among experts in this field that such membranes cannot withstand concentrated forces normal to their surfaces (cf. Novozhilov (1959), p. 100). That such curved surfaces should behave the same as flat ones in response to concentrated normal forces may be expected for, in the small vicinity of the force, the surface is effectively flat.

3. Concentrated moments on plates and shells

Classical bending theories for plates and shells admit concentrated forces in a reasonable manner. In particular, plates and shells do resist such forces, exhibiting stiffness, unlike the membrane. Bending moments at the point of load application are infinite, but this is not a serious shortcoming. One should expect infinite stresses under point loads.

The classical equation of transverse motion for thin, homogenous, isotropic, elastic plates is

$$D\nabla^4 w + \rho h \frac{\partial^2 w}{\partial t^2} = p, \tag{15}$$

where $D = Eh^3/12(1-v^2)$ is the flexural rigidity, $\nabla^4 = \nabla^2\nabla^2$, the biharmonic differential operator, and the other quantities are the same as those used in Eq. (1) for the membrane. For a static problem, the inertia term is discarded, for free vibrations, $p = 0$, and for dynamic response problems, including forced vibrations, all terms are retained.

Solutions abound for static plate problems with concentrated applied forces, especially for circular plates. For example, Timoshenko showed (Timoshenko and Woinowsky-Krieger, 1959, p. 69) that for a clamped circular plate of radius a, with a concentrated force (P) at its center, the deflection is

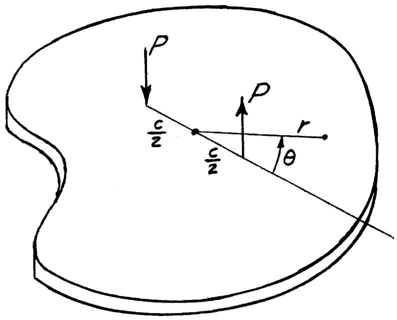

Fig. 4. Plate subjected to a couple, Pc.

$$w = \frac{P}{8\pi D} r^2 \ln r + \frac{P}{16\pi D}(a^2 - r^2) \tag{16}$$

resulting in a finite maximum deflection $w_{\max} = Pa^2/16\pi D$ at the plate center, $r = 0$, even though a logarithmic term appears in Eq. (16). Unlike the membrane result in Eq. (4), $\ln r$ is here multiplied by r^2. However, the bending moments within the plate involve $\mathrm{d}^2w/\mathrm{d}r^2$ and $(1/r)\mathrm{d}w/\mathrm{d}r$, which yield simple $\ln r$ terms in their expressions, and corresponding infinite bending stresses under the force.

Timoshenko also dealt later with the concentrated force in his classic work (Timoshenko and Woinowsky-Krieger, 1959, p. 325), but mainly to use it as a basis for generating a concentrated moment. One may first apply two equal, and oppositely directed, forces P to a plate, separated from each other by a distance c, to form a couple of magnitude Pc, as shown in Fig. 4. In terms of a *local* coordinate origin located at each force application point, the plate deflection due to each force is

$$w = \frac{P}{8\pi D} r_i^2 \ln\left(\frac{r_i}{d}\right), \tag{17}$$

where d is an arbitrary length. Superimposing the two solutions, and taking the limit as $c \to 0$, with $Pc = M_0$ remaining constant, one arrives at a representation of a concentrated moment of magnitude M_0. Timoshenko gave the resulting plate deflection as

$$w = \frac{M_0}{4\pi D} r \ln\frac{r}{d}\cos\theta. \tag{18}$$

But if one evaluates the slope, which is the tangent of the rotation angle, then

$$\frac{\mathrm{d}w}{\mathrm{d}r} = \frac{M_0}{4\pi D}\left(1 + \ln\frac{r}{d}\right)\cos\theta. \tag{19}$$

At the point of application ($r = 0$), the slope is seen to become infinite, which corresponds to a 90° angle of rotation, *regardless of magnitude*, M_0, of the applied moment. Thus, the plate has no rotational resistance to a concentrated moment. Timoshenko made no mention of this.

One will arrive at the same conclusion if, for example, a moment, M_0, is applied to a rigid central disk of a circular plate, as shown in Fig. 5. If M_0 is kept constant, and the ratio (b/a) of the disk radius to the plate is decreased, the rotation angle increases without bound as $b/a \to 0$. One can see this trend in Table 64 of Timoshenko's book, which shows results for this problem, although the range of the table is only for $0.5 \leqslant b/a \leqslant 0.8$.

Consequently, if one is calculating influence coefficients (or stiffness matrices) for plates, according to classical plate theory, one can do it for concentrated forces. But concentrated moments are meaningless.

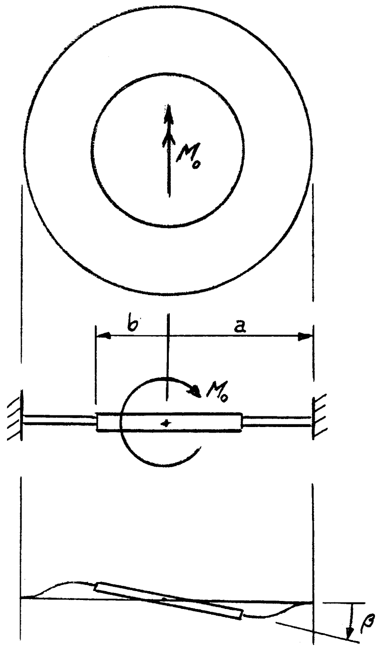

Fig. 5. Annular plate with bending moment, M_0, applied to a rigid interior disk of radius b.

The published literature is replete with examples of plates which have moments applied to points. In particular, problems are supposedly solved for plates which are *clamped* at points (not merely supported), either statically or dynamically loaded – in the case of free vibrations, the dynamic loading is inertial. “Clamped point” solutions are found for cases of either interior or boundary points, or both. Numerical results are reported for static deflections and free vibration frequencies, but they do not represent the situation claimed. The solutions are approximate, not exact, and therefore, result in the moments being applied along a finite length, instead of at a point.

The writer encountered this situation a few decades ago when he and a coworker attempted to obtain accurate results for the static deflection and bending moments in a circular plate, uniformly loaded by transverse pressure, with its boundary partially clamped (along a sector angle 2α) on opposite sides, and the remaining portion simply supported (Leissa and Clausen, 1967), as depicted in Fig. 6. The approximate boundary point least squares method (a generalization of point matching, or boundary collocation) was used. But as the clamping sector angle, 2α, diminished, the solution became unacceptably poor (large boundary residuals which should be zero).

The writer has also encountered published papers wherein results are claimed for the *buckling* of plates which are clamped at points. The same arguments apply here. That is, a point is capable of supplying a valid transverse concentrated force to constrain the displacement, thereby increasing the buckling loads, but it cannot generate a concentrated moment to restrain rotation.

Additional limitations are readily seen, as were elaborated upon in a bit more detail for membranes in Section 2, except that for plates they apply to concentrated moments and rotational constraints, instead of concentrated forces and translational constraints.

And the same arguments apply to classical shells; i.e., thin shells having both membrane and bending stiffness. They, too, cannot resist concentrated moments at either interior or boundary points, nor can point constraints resist rotations.

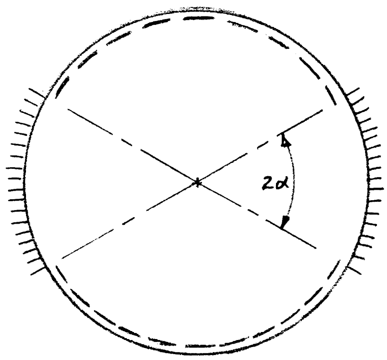

Fig. 6. Circular plate, partially clamped and partially simply supported along its periphery.

4. Sharp corner singularities in plates and shells

Every stress analyst knows that interior sharp corners will produce infinite stresses (i.e., singularities) there when loads are applied, and large stresses in the nearby surrounding region. This will occur for a plate subjected to static or dynamic inplane loads (the plane stress problem), or bending forces or moments, or for a shell. What does not seem to be generally known is that the singularities can also have significant effects upon the *global* behavior of the plate or shell; e.g. its static deflection, free vibration frequencies, forced dynamic response, or its critical buckling load. Or, to put it in other terms, if the stresses at the sharp corner are not properly accounted for, significant errors in the calculated global behavior may result.

Three examples of plates undergoing bending deformation, where such important singularities may arise, are seen in Fig. 7: (1) a cantilevered skew plate, clamped along one edge and free on the others; a circular sectorial plate, with two free radial edges forming a re-entrant sharp corner; and a parallelogram plate (shown as a rhombus) with all edges simply supported. Points where bending moment and stress singularities arise in each of these examples are marked in Fig. 7 with an "S".

When preparing to write Chapters 5–7 of his plate vibration monograph (Leissa, 1969), the writer found a large amount of published results for the free vibration frequencies and mode shapes of cantilevered skew plates, particularly for the cantilevered parallelogram shape. This seemed to be a consequence of the great need for data which could be used for subsequent flutter and vibration analyses of aircraft and missile wings or stabilizing surfaces. No exact solution of the free vibration problem is possible, and several approximate

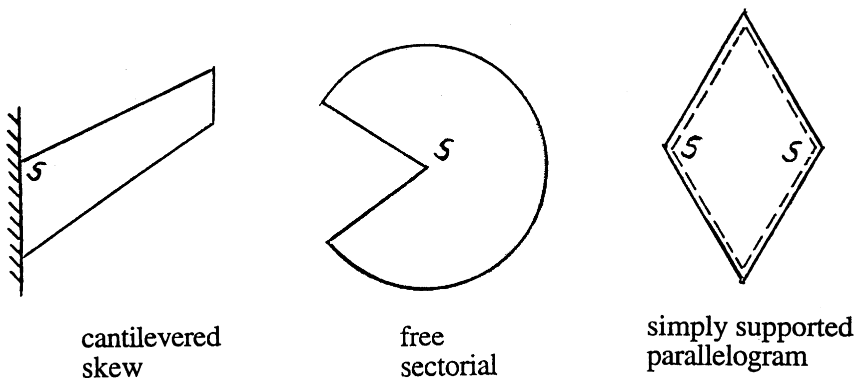

Fig. 7. Examples of plates having corner singularities.

methods were used, as described in the monograph. As the skew angle (or "sweep angle") is increased, it was found that not only were the frequencies changed greatly, but that significant differences were seen between the methods used in how large the changes were. At the obtuse corner where the clamped and free edges intersect, the bending and twisting moments are singular, the strength of the singularity increasing as the skew (and obtuse) angles increase. None of the analytical methods used gave any special consideration to the singularities, and the stresses in the corner resulting from the analyses were always finite.

The stress singularities in sharp corners can be represented exactly, as was shown in two classic papers by Williams (1952a,b). For each type of corner, there is an infinite set of functions which satisfy the boundary conditions exactly on both intersecting edges, which we call corner functions. Thus, if such functions are added to another global set of displacement functions, the corner stresses can be accounted for properly. This will be demonstrated below for free vibrations of the first two types of plates shown in Fig. 7. The Ritz method will be utilized. Mathematically complete sets of admissible displacement functions will be employed, which only need to satisfy the geometric (displacement and slope) boundary conditions. Such functions yield upper bound approximations to the frequencies, and the exact frequencies may be approached as closely as desired if sufficient terms are taken. However, a set of corner functions will be added, which accelerates the convergence exactly, because the stress singularity in the sharp corner is accounted for.

Fig. 8 shows a cantilever skew plate of parallelogram shape having sides of length a and b. Stress singularities occur at point O. For free vibrations, the transverse displacement, w, is sinusoidal in time:

$$w(\xi,\eta,t) = W(\chi,\eta)\sin\omega t, \tag{20}$$

where ω is a natural frequency. Further, the displacement form is taken as $W = W_\mathrm{p} + W_\mathrm{c}$, where W_p is a sum of algebraic polynomials,

$$W_\mathrm{p}(\xi,\eta) = \sum_{m=2}^{M}\sum_{n=0}^{N} A_{mn}\xi^m\eta^n \tag{21}$$

and W_c is a sum of corner functions,

$$W_\mathrm{c}(\xi,\eta) = \sum_{i=1}^{I} B_i W_i(\xi,\eta) + \sum_{j=1}^{j} C_j W_j^*(\xi,\eta), \tag{22}$$

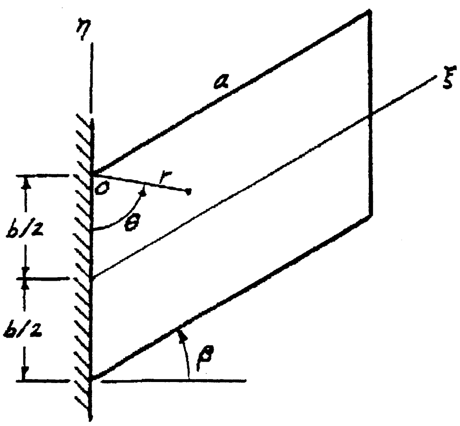

Fig. 8. Cantilever skew plate, with coordinates.

Table 2
Skew plate frequencies $\omega a^2 \cos^2\beta(\rho/D)^{1/2}$ for $a/b = 1$, $\beta = 60°$, $\nu = 0.3$

Mode number	Number of corner functions	Number of terms in polynomial (W_p)			
		3 × 3	5 × 5	6 × 6	7 × 7
	0	1.742	1.393	1.353	1.335
1	2	1.343	1.312	1.311	1.311
	10	1.312	1.311	1.311	1.311
	0	5.163	4.186	4.110	4.076
2	2	4.274	4.022	4.011	4.007
	10	4.019	4.007	4.006	4.006
	0	15.527	8.521	8.042	7.880
3	2	12.820	7.980	7.635	7.602
	10	7.810	7.625	7.593	7.591
	0	18.469	12.724	11.989	11.652
4	2	15.725	11.513	11.403	11.335
	10	11.505	11.400	11.336	11.325

where W_i and W_j^* are the real and imaginary parts of the complex eigenfunctions (Williams, 1952a) satisfying the clamped and free boundary conditions along the radial lines $\theta = 0$ and $\pi/2 + \beta$ (Fig. 8). Substituting Eqs. (21) and (22) into the proper strain energy and kinetic energy functionals (Leissa, 1969) and using the Ritz method, free vibration frequencies (eigenvalues) and mode shapes (eigenfunctions) are determined.

Table 2 shows nondimensional frequencies obtained for a rhombic ($a/b = 1$) plate with a 60° skew angle and Poisson's ratio, ν, of 0.3. Results are given using 9, 25, 36, or 49 terms in W_p, and 0, 2, or 10 terms in W_c. The frequencies are all upper bounds on the exact values. It is seen that poor convergence is obtained when only the algebraic polynomials are used, and that adding two corner functions improves the convergence considerably, because the corner stress singularities are then explicitly taken into account. For larger skew angles, the stress singularities become increasingly important, as one finds for $\beta = 75°$ (McGee et al., 1992). Using polynomials alone would, *in principle*, yield results which will converge properly to accurate values, but an enormous number of terms are needed, and round-off errors (ill conditioning) would destroy the accuracy before adequate convergence is achieved.

A completely free, circular plate of radius R is depicted in Fig. 9. A V-notch having its vertex a distance "c" from the center is cut into the plate. Transverse displacements are again assumed as $W = W_p + W_c$, where a product of algebraic and trigonometric polynomials is taken for W_p, and where free–free corner functions are used for W_c. The Ritz method is again used to solve the free vibration problem.

Nondimensional frequencies for a completely free circular plate with a 30° notch ($\alpha = 330°$) cut one-fourth of the way across the plate ($c/R = 0.5$) (Fig. 9 is drawn to these dimensions) are listed in Table 3. Convergence using various numbers of polynomial and corner function terms is observed. It is seen that poor convergence is obtained if insufficient corner functions are used. More details for this problem and its solution procedure may be found in the study by Leissa et al. (1993). For a very sharp notch or crack ($\alpha > 359°$ in Fig. 9), the corner functions were found to be especially important.

Various analyses for the free vibrations of the simply supported rhombic plate shown in Fig. 7 appear in the literature, some of which are summarized in the monograph (Leissa, 1969). Accurate frequencies were obtained by Huang et al. (1995) by the Ritz method, again using corner functions to represent the stress singularities in the obtuse corners. It was found that the use of corner functions in the obtuse corners to deal with the moment singularities there improved the results considerably, especially for large angles. In that situation, the obtuse corner dominates the problem, and the moment singularities are also stronger.

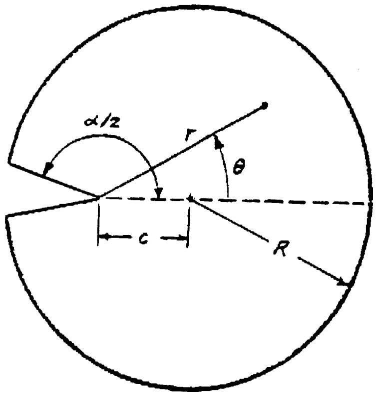

Fig. 9. Circular plate with V-notch.

Table 3
Frequencies $\omega R^2(\rho/D)^{1/2}$ for a completely free plate with a V-notch ($\alpha = 330°$, $c/R = 0.5$, $\nu = 0.3$)

Mode no. (symmetry class)	Number of corner functions	Number of terms in polynomial (W_p)			
		3 × 3	5 × 5	6 × 6	7 × 7
	0	5.320	5.318	5.315	5.312
1	1	5.064	5.021	5.021	5.006
(A)	5	4.959	4.949	4.942	4.937
	15	4.898	4.892	4.889	4.887
	0	5.488	5.484	5.478	5.471
2	1	5.317	5.310	5.305	5.302
(S)	5	5.304	5.300	5.298	5.297
	15	5.293	5.291	5.290	5.289
	0	9.030	9.022	9.014	9.005
3	1	8.802	8.794	8.789	8.786
(S)	5	8.789	8.785	8.782	8.780
	15	8.773	8.770	8.769	8.768
	0	12.326	12.317	12.307	12.298
4	1	11.155	11.073	11.004	10.949
(A)	5	10.785	10.745	10.718	10.701
	15	10.571	10.551	10.541	10.534

The writer knows of no application of the corner singularity functions to shell problems. However, it is expected that the singularity effects upon the global behavior of shells would be at least as important as those found for plates. For such configurations, both bending and membrane stresses would have singularities, and both the bending and plane stress singularity functions identified by Williams (1952a,b) should be used.

5. Concluding remarks

The primary objective of this paper was to identify situations in membranes, plates or shells where stress singularities may have strong effects upon the global behavior of the configuration. Indeed, in the case of concentrated forces acting upon flat or curved membranes, or concentrated moments acting upon plates or shells, the singularities render the theoretical problem meaningless. Of course, the physical problem does not exist either, for forces and moments must be applied to finite areas, or at least along lines of finite length, to be meaningful. Concentrated forces or moments would seem to be reasonable theoretical representations, for they are used successfully on beams or stretched strings. But these are one-dimensional models, whereas membranes, plates and shells are two-dimensional models.

The sharp corner effects on plate and shell behavior need to be taken into account in any analysis. How it was done with the Ritz method was described above. Finite elements are now the most popular method of structural analysis in use. But if the sharp corners cause stress singularities (e.g., external free corners do not), then elements should be used in those which include the proper singularities in their shape functions.

The global effects of singularities were demonstrated above for problems involving static deflections and free vibrations. Clearly, other situations such as forced vibrations, static buckling, dynamic instability, wave propagation, or general dynamic response would be similarly affected.

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